REMARKS ON NONDEGENERACY OF GROUND STATES FOR QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we answer affirmatively the problem proposed by A. Selvitella in his paper "Nondegenracy of the ground state for quasilinear Schrödinger Equations" (see Calc. Var. Partial Differ. Equ., **53** (2015), pp 349-364): every ground state of equation

$$-\Delta u - u\Delta|u|^2 + \omega u - |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^N$$

is nondegenerate for $1 < p < 3$, where $\omega > 0$ is a given constant and $N \geq 1$. We also derive further properties on the linear operator associated to ground states of above equation.

Keywords: Quasilinear Schrödinger equations; Ground states; Nondegeneracy

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Contents

1. Introduction and main result

Consider the quasilinear Schrödinger equation

$$i\partial_t U = -\Delta U - U\Delta|U|^2 - |U|^{p-1}U \qquad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$

where $U : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{C}$ is the wave function, i is the imaginary unit, $N \geq 1$ and $p > 1$. Equation (1.1) arises in various domains of physics, such as superfluid film equation in plasma physics. More physical background of equation (1.1) can be found in e.g. Colin et al. [5] and the references therein. Equation (1.1) has been studied extensively in the literature, see e.g. [3, 5, 11, 13, 19, 20] and the references therein. A special class of solutions to equation (1.1) that represent particles at rest is the so called standing waves, that is, solutions of the form

$$U(x,t) = e^{i\omega t}u(x),$$

where $\omega > 0$ is a given constant which stands for the time frequency, and $u : \mathbb{R}^N \to \mathbb{C}$ is a complex valued function that is independent of time $t \in \mathbb{R}_+$. It is elementary to verify that if $U(x,t) = e^{i\omega t}u(x)$ is a standing wave, then u solves the stationary equation

$$-\Delta u - u\Delta|u|^2 + \omega u - |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) is known [5] as the Euler-Lagrange equation of the energy functional $\mathcal{E}_\omega : \mathbb{X}_\mathbb{C} \to \mathbb{R}$ defined as

$$\mathcal{E}_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |u|^2|\nabla|u||^2 \mathrm{d}x + \frac{\omega}{2}\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\mathrm{d}x,$$

where $\mathbb{X}_\mathbb{C}$ is the function space given by

$$\mathbb{X}_\mathbb{C} = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2|\nabla|u||^2 \mathrm{d}x < \infty \right\}.$$

Here we assume throughout the paper that $1 < p < p_{\max}$ holds, where the critical exponent $p_{\max}$ is defined as

$$p_{\max} \equiv \begin{cases} \frac{3N+2}{N-2} & \text{if } N \geq 3 \\ \infty & \text{if } N = 1, 2. \end{cases} \tag{1.3}$$

Technically speaking, the condition that p be strictly less than $p_{\max}$ ensures that the power nonlinearity in equation (1.2) is $\mathbb{X}_\mathbb{C}$-subcritical. Indeed, by a simple application of Sobolev embedding theorems, we infer that $\mathbb{X}_\mathbb{C}$ is continuously embedded into $L^{p+1}(\mathbb{R}^N)$ for $1 < p < \infty$ if $N = 1, 2$ and $1 < p \leq (3N+2)/(N-2)$ if $N \geq 3$.

In view of the variational structures of equation (1.2), critical point theory has been devoted to find solutions for equation (1.2). Here, as in Colin et al. [5], a function $u \in \mathbb{X}_\mathbb{C}$ is said to be a solution to equation (1.2), if for any function $\phi \in C_0^\infty(\mathbb{R}^N)$, the space of smooth functions in $\mathbb{R}^N$ with compact support, there holds

$$\mathrm{Re}\int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla\bar{\phi} + \nabla|u|^2 \cdot \nabla(u\bar{\phi}) + \omega u\bar{\phi} - |u|^{p-1}u\bar{\phi} \right) \mathrm{d}x = 0$$

(here $\mathrm{Re}z$ is the real part of $z \in \mathbb{C}$). The existence of solutions to equation (1.2) is now well known, see e.g. [4, 5, 16, 17, 18] and the references therein.

In this paper, we consider ground state to equation (1.2). Following the convention of Colin et al. [5] (see also Selvitella [21, 22]), we say that a solution $u \in \mathbb{X}_\mathbb{C}$ to equation (1.2) is a *ground state*, if u satisfies

$$\mathcal{E}_\omega(u) = \inf \left\{ \mathcal{E}_\omega(v) : v \in \mathbb{X}_\mathbb{C} \text{ is a nontrivial solution to equation (1.2)} \right\}.$$

We remark that the notion of ground state here is different from that defined in [2, 6, 7, 12, 14]. We are concerned about the nondegeneracy (see below) of ground states. Before proceeding further, let us summarize the existence result of ground states to equation (1.2) together with a list of basic properties for later use.

Theorem 1.1. *Assume that $1 < p < p_{\max}$ with $p_{\max}$ defined as in (1.3). Then for any given constant $\omega > 0$, there exists a ground state to equation (1.2). Moreover, for any ground state $u \in \mathbb{X}_\mathbb{C}$ to equation (1.2), there exist a constant $\theta \in \mathbb{R}$, a decreasing positive function $v : [0, \infty) \to (0, \infty)$ and a point $x_0 \in \mathbb{R}^N$ such that u is of the form*

$$u(x) = e^{i\theta}v(|x - x_0|) \qquad \textit{for } x \in \mathbb{R}^N.$$

Furthermore, the following properties hold.

(1) (Smoothness) $u \in C^\infty(\mathbb{R}^N)$.

(2) (Decay) For any multi-index $\alpha \in \mathbb{N}^N$ *with* $|\alpha| \geq 0$, *there exist positive constants* $C_\alpha > 0$ *and* $\delta_\alpha > 0$ *such that*

$$|\partial^\alpha u(x)| \leq C_\alpha \exp(-\delta_\alpha |x|) \qquad \text{for all } x \in \mathbb{R}^N.$$

(3) (Uniqueness) In the case $N = 1$, *the ground states to equation (1.2) is unique up to phase and translation. In particular, there exists a unique positive even ground state for equation (1.2).*

For a complete proof of Theorem 1.1, we refer to Colin et al. [5] and Selvitella [21].

In this paper, our aim is to study nondegeneracy of ground states for equation (1.2). The motivation comes from the fact that the nondegeneracy of ground states for equation (1.2) plays an important role when studying the existence of concentrating solutions in the semiclassical regime. We refer the readers to Selvitella [22] for more applications of nondegeneracy results. We also follow the convention of Selvitella [22] (see also Ambrosetti and Malchiodi [1]) and define nondegeneracy of ground states for equation (1.2) as follows.

Definition 1.2. *Let* $u \in \mathbb{X}_{\mathbb{C}}$ *be a ground state of equation (1.2). We say that* u *is nondegenerate if the following properties hold:*

(1) (ND) $\mathrm{Ker}\mathcal{E}''_\omega(u) = \mathrm{span}\,\{iu, \partial_{x_1} u, \cdots, \partial_{x_N} u\}$;

(2) (Fr) $\mathcal{E}''_\omega(u)$ *is an index 0 Fredholm map.*

The first result on nondegeneracy of ground states for equation (1.2) was obtained by Selvitella [21] in a perturbative setting, where uniqueness of ground states for equation (1.2) was also considered. In his quite recent paper [22], Selvitella proved, under the assumption

$$p \geq 3,$$

that every ground state of equation (1.2) is nondegenerate in the sense of Definition 1.2 above, see Theorem 1.2 of [22]. Selvitella also commented (see Remark 1.3 of [22]) that his nondegeneracy result could also be true for the case $1 < p < 3$. However, his approach can not handle this case. In this paper, we give an affirmative answer to his question. We obtain the following result.

Theorem 1.3. *For* $1 < p < 3$, *every ground state of equation (1.2) is nondegenerate in the sense of Definition 1.2 above.*

We remark that our argument is applicable to the whole range $1 < p < p_{\max}$.

As already remarked by Selvitella (see Remark 1.3 of [22]), except Proposition 3.10 of [22] that requires him to assume $p \geq 3$, all the rest of his arguments can be applied to the whole range $1 < p < p_{\max}$ to prove Theorem 1.3. So in this paper, we will follow the line of Selvitella [22] to prove Theorem 1.3. However, since his approach can not handle the whole range $1 < p < p_{\max}$, we will apply a different idea from that of Selvitella [22]. Precisely, let u be a positive radial ground state of equation (1.2). Define the linear operator $\mathcal{L}_+$ associated to u by

$$\mathcal{L}_+ \eta = -\Delta \eta - 2u\Delta(u\eta) + \omega\eta - (\Delta u^2 + pu^{p-1})\eta.$$

We note that $\mathcal{L}_+$ is a self-adjoint operator acting on $L^2(\mathbb{R}^N)$ with form domain $\mathbb{X}_{\mathbb{C}}$ and operator domain $H^2(\mathbb{R}^N)$. It turns out that the key to prove Theorem 1.3 is to show that $\mathcal{L}_+$ satisfies

$$\mathrm{Ker}\mathcal{L}_+ = \mathrm{span}\left\{\partial_{x_1}u, \cdots, \partial_{x_N}u\right\}. \tag{1.4}$$

In the approach of Selvitella [22] to (1.4), ordinary differential equation analysis plays a central role, in which the assumption $p \geq 3$ is required. To prove (1.4) for p in the whole range $1 < p < p_{\max}$, we will use a spectrum analysis to the operator $\mathcal{L}_+$. In this way, we obtain deeper results on the operator $\mathcal{L}_+$ than (1.4).

Our idea comes from the spectrum analysis of Chang et al. [2], even through we can not use their refined arguments to derive (1.4) directly. Roughly speaking, Chang et al. [2] considered the following problem. Let Q the unique positive radial solution in $H^1(\mathbb{R}^N)$ to the equation

$$-\Delta Q + \omega Q - Q^q = 0 \qquad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $1 < q < (N+2)/(N-2)$. Here we assume $N \geq 3$ for simplicity. Chang et al. [2] studied the spectrum of the linear operator $\mathcal{A}_+$ around Q given by

$$\mathcal{A}_+\eta = -\Delta\eta + \omega\eta - qQ^{q-1}\eta, \tag{1.6}$$

acting on $L^2(\mathbb{R}^N)$ with form domain $H^1(\mathbb{R}^N)$ and operator domain $H^2(\mathbb{R}^N)$. For the importance of the spectrum of $\mathcal{A}_+$, we refer the readers to Chang et al. [2]. We remark that Chang et al. [2] studied far more than $\mathcal{A}_+$ in their work. We will give a brief comparison between the two self-adjoint operators $\mathcal{A}_+$ and $\mathcal{L}_+$ below. We also refer the readers to [6, 7, 14] for spectrum analysis for linearized operators around ground states of nonlocal problems.

It seems that the spectrum $\sigma(\mathcal{L}_+)$ of $\mathcal{L}_+$ has not been studied in the literature. According to the analysis in next section, we obtain the following properties for the spectrum of $\mathcal{L}_+$:

(1) the continuous spectrum of $\mathcal{L}_+$ is contained in $[\omega, \infty)$ (see Lemma 2.4);

(2) $\inf\sigma(\mathcal{L}_+) < 0$ is the first eigenvalue of $\mathcal{L}_+$ and is simple (see Lemma 2.5);

(3) as a consequence of (1) and (2), 0 belongs to the discrete spectrum of $\mathcal{L}_+$ and is not the first eigenvalue of $\mathcal{L}_+$.

We will give the proof of Theorem 1.3 together with above properties of $\sigma(\mathcal{L}_+)$ in the next section. Before we close this section, let us address some differences between the two self-adjoint operators $\mathcal{A}_+$ defined as in (1.6) and $\mathcal{L}_+$.

First we point out that the second property (2) is not obvious. In fact, even the fact $\inf\sigma(\mathcal{L}_+) < 0$ is not obvious. For the operator $\mathcal{A}_+$, a simple observation gives that

$$\langle \mathcal{A}_+Q, Q\rangle = -(q-1)\int_{\mathbb{R}^N} Q^{q+1}\mathrm{d}x < 0,$$

which implies $\inf\sigma(A_+) < 0$. Furthermore, it is standard (see e.g. Lieb and Loss [15]) to show that $\inf\sigma(\mathcal{A}_+)$ is the first eigenvalue of $\mathcal{A}_+$ and is simple. However, in our case, a direct calculation gives us

$$\langle \mathcal{L}_+u, u\rangle = 8\int_{\mathbb{R}^N} u^2|\nabla u|^2\mathrm{d}x - (p-1)\int_{\mathbb{R}^N} u^{p+1}\mathrm{d}x.$$

We will confirm that $\langle \mathcal{L}_+ u, u\rangle < 0$ in the case $N = 2$, based on a Pohozaev type identity, see the end of next section. While in higher dimensions $N \geq 3$, it is still not clear from above expression whether $\langle \mathcal{L}_+ u, u\rangle$ is negative or not. Hence we can not infer that $\inf \sigma(\mathcal{L}_+) < 0$ holds by such a simple observation as above. Similarly, due to the presence of the quasilinear term $-2u\Delta(u\eta)$ in $\mathcal{L}_+$, it is not obvious as well that a nonnegative eigenfunction of $\mathcal{L}_+$ is in fact positive everywhere.

Second, we point out that we do not known whether 0 is the second eigenvalue of $\mathcal{L}_+$ or not. Then we can not give exact estimates on the numbers of nodal domains of radial functions v with $v \in \mathrm{Ker}\mathcal{L}_+$. Thus we can not use the arguments of Chang et al. [2] directly (see the proof of Lemma 2.1 of Chang et al. [2]). As to the operator $\mathcal{A}_+$, it is known (see e.g. Chang et al. [2]) that 0 is the second eigenvalue of $\mathcal{A}_+$. This is due to fact that, by uniqueness, Q is also a minimizer (up to rescaling) of the 'Weinstein' functional

$$W(f) = \frac{\|\nabla f\|_2^{(q-1)N/2}\|f\|_2^{(N+2-(N-2)q)/2}}{\|f\|_{q+1}^{q+1}}, \qquad f \in H^1(\mathbb{R}^N), f \not\equiv 0.$$

In our case, except the case $N = 1$ (see Colin et al. [5]), the uniqueness of positive solutions to equation (1.2) is unknown in general when $N \geq 2$. In fact, even the uniqueness of ground states to equation (1.2) is unknown in general when $N \geq 2$. Some partial results on uniqueness were obtained in the literature. Since this is out of the scope of this paper, we refer the interested readers to Selvitella [21, 22] and the references therein.

Our nations are standard. We write $\mathbb{R}_+ = (0, \infty)$ and denote $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of nonnegative integers. For any $1 \leq s \leq \infty$, $L^s(\mathbb{R}^N)$ is the Banach space of complex valued Lebesgue measurable functions u such that the norm

$$\|u\|_s = \begin{cases} \left(\int_{\mathbb{R}^N} |u|^s \mathrm{d}x\right)^{\frac{1}{s}} & \text{if } 1 \leq s < \infty \\ \operatorname{esssup}_{\mathbb{R}^N} |u| & \text{if } s = \infty \end{cases}$$

is finite. A function u belongs to the Sobolev space $H^k(\mathbb{R}^N)$ ($k \in \mathbb{N}$) if $u \in L^2(\mathbb{R}^N)$ and its weak partial derivatives up to order k also belong to $L^2(\mathbb{R}^N)$. We equip $H^k(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^k} = \sum_{\alpha \in \mathbb{N}^N, |\alpha| \leq k} \|\partial^\alpha u\|_2.$$

For the properties of the Sobolev functions, we refer to the monograph [23]. By abuse of notation, we write $f(x) = f(r)$ with $r = |x|$ whenever f is a radially symmetric function in $\mathbb{R}^N$.

2. Proof of main result

In this section we prove Theorem 1.3. Since we deal with the same problem as that of Selvitella [22], we will follow the line of Selvitella [22]. Similar lines can also be found in e.g. [2, 6, 7, 14] and the monograph [1].

2.1. **Proof of Theorem 1.3.** Let $u \in \mathbb{X}_{\mathbb{C}}$ be an arbitrary ground state for equation (1.2). By Definition 1.2, to prove Theorem 1.3 we have to show that $\mathcal{E}''_\omega(u)$ satisfies property (ND) and property (Fr). The property (Fr) can be proved by the same argument as that

of Selvitella [22], since which is applicable to the whole range $1 < p < p_{\max}$. So we omit the details.

We focus on the proof of the property (ND), that is, we prove in the following that

$$\mathrm{Ker}\mathcal{E}''_\omega(u) = \mathrm{span}\left\{iu, \partial_{x_1}u, \cdots, \partial_{x_N}u\right\}. \tag{2.1}$$

By Theorem 1.1, every ground state of equation (1.2) can be regarded as a positive, radial and symmetric-decreasing ground state. Hence we assume in the sequel that $u = u(|x|) > 0$ is a positive, radial and symmetric-decreasing ground state for equation (1.2). We also assume $N \geq 2$ in the sequel. In the case $N = 1$ the proof of (2.1) is similar and even simpler. Then the linearized operator $\mathcal{E}''_\omega(u)$ is giving by

$$\mathcal{E}''_\omega(u)\xi = -\Delta\xi - 2u\Delta\left(u\mathrm{Re}\xi\right) + \omega\xi - \left(\Delta u^2\right)\xi - (p-1)u^{p-1}\mathrm{Re}\xi - u^{p-1}\xi$$

acting on $L^2(\mathbb{R}^N)$ with form domain $\mathbb{X}_{\mathbb{C}}$ and operator domain $H^2(\mathbb{R}^N)$.

Note that $\mathcal{E}''_\omega(u)$ is not even $\mathbb{C}$-linear. To overcome this difficulty, it is preferable to introduce the linear operator $\mathcal{L}_+$ given by

$$\mathcal{L}_+\eta = -\Delta\eta - 2u\Delta(u\eta) + \omega\eta - (\Delta u^2 + pu^{p-1})\eta, \tag{2.2}$$

acting on $L^2(\mathbb{R}^N)$ with form domain $\mathbb{X}_{\mathbb{C}}$ and operator domain $H^2(\mathbb{R}^N)$, and the linear operator $\mathcal{L}_-$ given by

$$\mathcal{L}_-\zeta = -\Delta\zeta + \omega\zeta - (\Delta u^2 + u^{p-1})\zeta$$

acting on $L^2(\mathbb{R}^N)$ with form domain $H^1(\mathbb{R}^N)$ and operator domain $H^2(\mathbb{R}^N)$. Then for any $\xi \in H^2(\mathbb{R}^N)$ we obtain

$$\mathcal{E}''_\omega(u)\xi = \mathcal{L}_+\mathrm{Re}\xi + i\mathcal{L}_-\mathrm{Im}\xi$$

(here $\mathrm{Im}z$ is the imaginary part of $z \in \mathbb{C}$). Therefore, to prove (2.1), it is sufficient to prove the following result.

Proposition 2.1. *Let $\mathcal{L}_+$ and $\mathcal{L}_-$ be defined as above. We have that*

$$\mathrm{Ker}\mathcal{L}_+ = \mathrm{span}\left\{\partial_{x_1}u, \cdots, \partial_{x_N}u\right\} \tag{2.3}$$

and

$$\mathrm{Ker}\mathcal{L}_- = \mathrm{span}\left\{u\right\}. \tag{2.4}$$

Proof. First we prove (2.4). The proof is standard. In fact, we can use the argument of Selvitella [22] since which is applicable to p in the whole range of $1 < p < p_{\max}$. We give a proof here for the reader's convenience.

First we use spherical harmonics to decompose functions $v \in H^j(\mathbb{R}^N)$ for $j \in \mathbb{N}$. Denote by $-\Delta_{\mathbb{S}^{N-1}}$ the Laplacian-Beltrami operator on the unit $N-1$ dimensional sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Write

$$M_k = \frac{(N+k-1)!}{(N-1)!k!} \quad \forall\, k \geq 0, \quad \text{and} \quad M_k = 0 \quad \forall\, k < 0.$$

Denote by $Y_{k,l}$, $k = 0, 1, \ldots$ and $1 \leq l \leq M_k - M_{k-2}$, the spherical harmonics such that

$$-\Delta_{\mathbb{S}^{N-1}}Y_{k,l} = \lambda_k Y_{k,l}$$

for all $k = 0, 1, \ldots$ and $1 \leq l \leq M_k - M_{k-2}$, where

$$\lambda_k = k(N+k-2) \qquad \forall\, k \geq 0$$

are eigenvalues of $-\Delta_{\mathbb{S}^{N-1}}$ with multiplicities $M_k - M_{k-2}$. In particular, we deduce that $\lambda_0 = 0$ is of multiplicity 1 with $Y_{0,1} = 1$, and $\lambda_1 = N-1$ is of multiplicity N with $Y_{1,l} = x_l/|x|$ for $1 \le l \le N$.

Then for any function $v \in H^j(\mathbb{R}^N)$, we have

$$v(x) = v(r\Omega) = \sum_{k=0}^{\infty} v_k(r) Y_k(\Omega)$$

with $r = |x|$ and $\Omega = x/|x|$, where

$$v_k(r) = \int_{\mathbb{S}^{N-1}} v(r\Omega) Y_k(\Omega) \mathrm{d}\Omega \qquad \forall\, k \ge 0. \tag{2.5}$$

Note that $v_k \in H^j(\mathbb{R}_+, r^{N-1}\mathrm{d}r)$ holds for all $k \ge 0$ since $v \in H^j(\mathbb{R}^N)$.

Next, apply above decomposition to any function $v \in H^1(\mathbb{R}^N)$. We conclude that $\mathcal{L}_- v = 0$ if and only if

$$\mathcal{L}_{-,k} v_k \equiv -v_k'' - \frac{N-1}{r} v_k' + \frac{\lambda_k}{r^2} v_k + \omega v_k - (\Delta u^2 + u^{p-1}) v_k = 0$$

for all $k \ge 0$, where v_k is defined as in (2.5). Note that $\mathcal{L}_{-,k}$ is a self-adjoint operator acting on $L^2(\mathbb{R}_+, r^{N-1}\mathrm{d}r)$ for all $k \in \mathbb{N}$.

First we consider $k = 0$. In this case we have $\lambda_0 = 0$. By a direct computation, we obtain that $\mathcal{L}_{-,0} u = 0$. Since $u(r) > 0$ for all $r > 0$, we conclude in a standard way that u is the first eigenfunction and 0 is the first simple eigenvalue of $\mathcal{L}_{-,0}$. Thus we have

$$\mathrm{Ker}\mathcal{L}_{-,0} = \mathrm{span}\{u\}. \tag{2.6}$$

Next consider $k \ge 1$. We claim that for all $k \ge 1$, there holds

$$\mathrm{Ker}\mathcal{L}_{-,k} = \{0\}. \tag{2.7}$$

Indeed, since $\lambda_k > 0$, we deduce that $\mathcal{L}_{-,k} > \mathcal{L}_{-,0}$ holds in the sense of quadratic form, which implies that $\mathcal{L}_{-,k} w = 0$ if and only if $w \equiv 0$. This proves the claim.

Finally, we infer from (2.6) and (2.7) that (2.4) holds. This finishes the proof of (2.4).

It remains to prove (2.3). We still use spherical harmonic as above. Then $\mathcal{L}_+ v = 0$ for $v \in \mathbb{X}_{\mathbb{C}}(\mathbb{R}^N)$ if and only if for all $k = 0, 1, \ldots,$ we have

$$\begin{aligned} \mathcal{L}_{+,k} v_k \equiv & -(1+2u^2)\left(v_k'' + \frac{N-1}{r} v_k' - \frac{\lambda_k}{r^2} v_k\right) - 4uu'v_k' + \omega v_k \\ & - (2u\Delta u + \Delta u^2 + pu^{p-1}) v_k = 0. \end{aligned} \tag{2.8}$$

For a detailed calculation of $\mathcal{L}_{+,k}$, we refer to Selvitella [22]. Note the fact that

$$\partial_{x_l} u = u'(|x|) \frac{x_l}{|x|} = u'(r) Y_{1,l} \qquad \text{for } 1 \le l \le N.$$

Thus to prove (2.3), it is sufficient to prove that

$$\mathcal{L}_{+,0} v_0 = 0 \qquad \text{if and only if } v_0 \equiv 0, \tag{2.9}$$

and that

$$\mathcal{L}_{+,1} v_1 = 0 \qquad \text{if and only if } v_1 \in \mathrm{span}\left\{u'\right\}, \tag{2.10}$$

and that

$$\mathcal{L}_{+,k}v_k = 0 \qquad \text{if and only if } v_k \equiv 0 \tag{2.11}$$

for all $k \geq 2$.

(2.10) and (2.11) can be proved in the same way as that of (2.6) and (2.7). Consider $k = 1$. In this case we have $\lambda_1 = N - 1$. We deduce from $\mathcal{L}_+\partial_{x_1}u = 0$ that $\mathcal{L}_{+,1}u' = 0$. Since $u'(r) < 0$ for all $r > 0$, we conclude in a standard way that u' is the first eigenfunction and 0 is the first simple eigenvalue of $\mathcal{L}_{+,1}$. This proves (2.10). To conclude (2.11), it is enough to notice that $\mathcal{L}_{+,k} > \mathcal{L}_{+,1}$ for any $k > 1$. This proves (2.11).

We leave the proof of (2.9) in the next subsection. The proof of Proposition 2.1 is complete. Thus the proof of Theorem 1.3 is complete. □

We remark that (2.4) can be proved in a more compact way. Indeed, note that $\mathcal{L}_-u = 0$ since u solves equation (1.2). Thus u is an eigenfunction of $\mathcal{L}_-$ with eigenvalue 0. Moreover, recall that u is a positive eigenfunction. We can conclude in a standard way that 0 is the first eigenvalue of $\mathcal{L}_-$ and is simple. Hence $\mathrm{Ker}\mathcal{L}_- = \mathrm{span}\{u\}$. See similar discussions in Chang et al. [2].

2.2. **Proof of (2.9).** Let us first briefly review the proof of (2.9) of Selvitella [22]. Suppose that v_0 belongs to $L^2(\mathbb{R}_+, r^{N-1}\mathrm{d}r)$, $v_0 \not\equiv 0$ and satisfies $\mathcal{L}_{+,0}v_0 = 0$. His proof (see Lemma 4.4 of Selvitella [22]) contains two ingredients. First he proved that $v_0(r)$ changes sign at least once for $r > 0$, and then by the disconjugacy interval argument of Kwong [12] he deduced that $v_0(r)$ is unbounded for $r > 0$ sufficiently large, which contradicts to $v_0 \in L^2(\mathbb{R}_+, r^{N-1}\mathrm{d}r)$. In this way Selvitella [22] proved (2.9). To prove that v_0 changes sign at least once on $\mathbb{R}_+$, Selvitella [22] used an ordinary differential equation analysis, in which the assumption $p \geq 3$ is needed (see Section 3 of Selvitella [22]). While the disconjugacy interval argument applies to the whole range $1 < p < p_{\max}$.

Taking into account above review, we infer that (2.9) can be deduced from the following result together with the disconjugacy interval argument as that of Kwong [12] and Selvitella [22].

Proposition 2.2. *Let $\mathcal{L}_{+,0}$ be defined as in (2.8) with $k = 0$. Suppose that v belongs to $L^2(\mathbb{R}_+, r^{N-1}\mathrm{d}r)$, $v \not\equiv 0$ and satisfies $\mathcal{L}_{+,0}v = 0$. Then $v(r)$ changes sign at least once for $r > 0$.*

Proposition 2.2 can be viewed as a substitute of Proposition 3.10 of Selvitella [22]. We use a spectrum analysis to prove Proposition 2.2.

First we note that $\mathcal{L}_{+,0}$ is the restriction of $\mathcal{L}_+$ on the sector $L^2_{\mathrm{rad}}(\mathbb{R}^N)$, the subspace of radial functions in $L^2(\mathbb{R}^N)$. Indeed, for any $v \in L^2_{\mathrm{rad}}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\mathcal{L}_+v &= -\Delta v - 2u\Delta(uv) + \omega v - (\Delta u^2 + pu^{p-1})v \\
&= -(1 + 2u^2)\left(v'' + \frac{N-1}{r}v'\right) - 4uu'v' + \omega v - (2u\Delta u + \Delta u^2 + pu^{p-1})v \\
&= \mathcal{L}_{+,0}v
\end{aligned}$$

since $\lambda_0 = 0$. Thus we immediately find the following result which is equivalent to Proposition 2.2.

Proposition 2.3. *Suppose that* $v \in \mathrm{Ker}\mathcal{L}_+ \cap L^2_{\mathrm{rad}}(\mathbb{R}^N)$ *is a nontrivial function. Then* $v(x) = v(r)$ *with* $r = |x|$ *changes sign at least once for* $r > 0$.

The idea to prove Proposition 2.3 is as follows. Note that 0 belongs to the spectrum $\sigma(\mathcal{L}_+)$ of $\mathcal{L}_+$, since it is straightforward to verify that

$$\mathrm{span}\,\{\partial_{x_1} u, \cdots, \partial_{x_N} u\} \subset \mathrm{Ker}\mathcal{L}_+.$$

We will show that 0 belongs to the discrete spectrum $\sigma_{\mathrm{disc}}(\mathcal{L}_+)$ of $\mathcal{L}_+$, that is, 0 is an isolated eigenvalue of $\mathcal{L}_+$ and the corresponding eigenfunction space is finite dimensional. We also show that 0 is not the first eigenvalue of $\mathcal{L}_+$. Then we have $\int_{\mathbb{R}^N} v e_1 \mathrm{d}x = 0$, where e_1 is the first eigenfunction of $\mathcal{L}_+$. This fact will imply that $v = v(r)$ changes sign for $r > 0$, once we prove that e_1 does not change sign in $\mathbb{R}^N$.

It is easy to verify that $\mathcal{L}_+$ is a self-adjoint operator acting on $L^2(\mathbb{R}^N)$ with form domain $\mathbb{X}_{\mathbb{C}}$ and domain $H^2(\mathbb{R}^N)$. Hence we have $\sigma(\mathcal{L}_+) \subset \mathbb{R}$. Furthermore, by Weyl's theorem (see Theorem 7.2 of Hislop and Sigal [10]) we have $\sigma(\mathcal{L}_+) = \sigma_{\mathrm{disc}}(\mathcal{L}_+) \cup \sigma_{\mathrm{cont}}(\mathcal{L}_+)$, and $\sigma_{\mathrm{disc}}(\mathcal{L}_+) \cap \sigma_{\mathrm{cont}}(\mathcal{L}_+) = \emptyset$, where $\sigma_{\mathrm{cont}}(\mathcal{L}_+)$ denotes the continuous spectrum of $\mathcal{L}_+$. Let us now start the proof of Proposition 2.3 with an estimate on $\sigma_{\mathrm{cont}}(\mathcal{L}_+)$. Recall that a constant λ belongs to $\sigma_{\mathrm{cont}}(\mathcal{L}_+)$ if and only if there exists a sequence $\phi_n \in H^2(\mathbb{R}^N)$, $n = 1, 2, \ldots$, such that

$$\|\mathcal{L}_+\phi_n - \lambda\phi_n\|_2 \to 0 \qquad \text{as } n \to \infty, \text{ and} \tag{2.12}$$

$$\|\phi_n\|_2 = 1 \qquad \text{for all } n \in \mathbb{N}, \text{ and} \tag{2.13}$$

$$\phi_n \rightharpoonup 0 \qquad \text{weakly in } L^2(\mathbb{R}^N) \text{ as } n \to \infty. \tag{2.14}$$

Lemma 2.4. *We have* $\sigma_{\mathrm{cont}}(\mathcal{L}_+) \subset [\omega, \infty)$.

Proof. Since $\mathcal{L}_+$ is self-adjoint, we have $\sigma(\mathcal{L}_+) \subset \mathbb{R}$. So it is sufficient to prove that if $\lambda < \omega$, then $\lambda \notin \sigma_{\mathrm{cont}}(\mathcal{L}_+)$. We argue by contradiction. Suppose, on the contrary, that $\lambda < \omega$ is a real number and $\lambda \in \sigma_{\mathrm{cont}}(\mathcal{L}_+)$. Then there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset H^2(\mathbb{R}^N)$ such that (2.12)-(2.14) hold. We claim that, up to a subsequence,

$$\phi_n \to 0 \qquad \text{strongly in } L^2(\mathbb{R}^N). \tag{2.15}$$

Then we reach to a contradiction to (2.13) and Lemma 2.4 is proved.

We prove (2.15) as follows. Note that $\Delta u^2 + pu^{p-1}$ is bounded in $\mathbb{R}^N$ by Theorem 1.1. Thus we obtain that

$$\sup_n \int_{\mathbb{R}^N} (\omega - \lambda + |\Delta u^2 + pu^{p-1}|)|\phi_n|^2 \mathrm{d}x < \infty.$$

On the other hand, we have

$$\begin{aligned} o(1) &= \langle (\mathcal{L}_+ - \lambda)\phi_n, \phi_n \rangle \\ &= \int_{\mathbb{R}^N} \left(|\nabla \phi_n|^2 + |\nabla(u\phi_n)|^2 + (\omega - \lambda - \Delta u^2 - pu^{p-1})|\phi_n|^2 \right) \mathrm{d}x. \end{aligned} \tag{2.16}$$

The first equality of above follows from (2.12) and (2.13). Therefore we derive directly from (2.16) that $|\nabla \phi_n| \in L^2(\mathbb{R}^N)$ is bounded uniformly for all $n \in \mathbb{N}$. Hence $\phi_n \in H^1(\mathbb{R}^N)$

is bounded uniformly for all n in view of (2.13). In particular, we deduce, after possibly passing to a subsequence, that

$$\phi_n \to 0 \qquad \text{strongly in } L^2_{\rm loc}(\mathbb{R}^N). \tag{2.17}$$

Next we recall that the function $\Delta u^2 + pu^{p-1}$ decays exponentially to zero at infinity by Theorem 1.1. Combining this fact together with (2.17) gives us that

$$\int_{\mathbb{R}^N} |\Delta u^2 + pu^{p-1}||\phi_n|^2 \mathrm{d}x \to 0 \tag{2.18}$$

as $n \to \infty$. Combining (2.18) with (2.16) and recalling that $\omega > \lambda$, we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\phi_n|^2 \mathrm{d}x = 0,$$

which contradicts to the assumption (2.13). The proof of Lemma 2.4 is complete. □

A direct consequence of Lemma 2.4 is that $0 \in \sigma_{\rm disc}(\mathcal{L}_+)$. Lemma 2.4 also allows us to derive a variational characterization for eigenvalues of $\mathcal{L}_+$ that are below the infimum of $\sigma_{\rm cont}(\mathcal{L}_+)$. Indeed, suppose that we have eigenvalues

$$\inf \sigma(\mathcal{L}_+) \equiv \mu_1 \le \mu_2 \le \cdots \le \mu_n < \inf \sigma_{\rm cont}(\mathcal{L}_+).$$

The fact $\mu_1 > -\infty$ follows easily from the elementary estimate

$$\inf_{\xi\in\mathbb{X}_{\mathbb{C}}, \|\xi\|_2=1} \langle \mathcal{L}_+\xi, \xi\rangle > -\infty.$$

Then we have

$$\mu_1 = \inf\left\{\langle \mathcal{L}_+\xi, \xi\rangle : \xi \in \mathbb{X}_{\mathbb{C}}, \|\xi\|_2 = 1\right\}.$$

Denoting by W_k, $2 \le k \le n$, the linear space spanned by the first $n-1$ eigenfunctions corresponding to $\mu_1, \ldots, \mu_{n-1}$, we have by induction

$$\mu_n = \inf\left\{\langle \mathcal{L}_+\xi, \xi\rangle : \xi \in \mathbb{X}_{\mathbb{C}}, \|\xi\|_2 = 1, \text{ and } \xi \perp W_n\right\}.$$

Furthermore, for any function $\xi \in \mathbb{X}_{\mathbb{C}}$ with $\|\xi\|_2 = 1$, $\xi \perp W_n$ and $\langle \mathcal{L}_+\xi, \xi\rangle = \mu_n$, ξ is a linear combination of eigenfunctions corresponding to μ_n.

Next we prove that $\mathcal{L}_+$ satisfies Perron-Frobenius property. That is, if $\inf \sigma(\mathcal{L}_+)$ is an eigenvalue, then it is simple and the corresponding eigenfunction can be chosen strictly positive. In the case of equation (1.5), it is easy to verify that the operator $\mathcal{A}_+$ satisfies the Perron-Frobenius property as follows. Let $\xi_1 \in L^2(\mathbb{R}^N)$ is an eigenfunction of $\mathcal{A}_+$ with eigenvalue $a_1 = \inf \sigma(\mathcal{A}_+) < 0$. Then ξ_1 solves equation

$$-\Delta\xi + (\omega - a_1)\xi_1 = qQ^{q-1}\xi_1.$$

Since above equation is linear, we can assume that $\xi_{1,+} = \max\{\xi_1, 0\} \not\equiv 0$. Then we obtain, by multiplying above equation by $\xi_{1,+}$, that $\langle \mathcal{A}_+\xi_{1,+}, \xi_{1,+}\rangle = 0$, which implies that $\xi_{1,+}$ is also an eigenfunction of $\mathcal{A}_+$ with eigenvalue a_1. Therefore, $\xi_{1,+}$ satisfies the equation of ξ_1 as well. Finally, note that $\omega - a_1 > 0$. Thus we have

$$\xi_{1,+} = \frac{1}{-\Delta + (\omega - a_1)} qQ^{q-1}\xi_{1,+}.$$

Since $Q > 0$ for all $x \in \mathbb{R}^N$ and since the integral kernel of the operator $\frac{1}{-\Delta+(\omega-a_1)}$ is positive away from the origin, we conclude from above formula that $\xi_{1,+}(x) > 0$ for all

$x \in \mathbb{R}^N$. Thus $\xi_1 = \xi_{1,+}$ is strictly positive in $\mathbb{R}^N$. This shows that $\mathcal{A}_+$ satisfies the Perron-Frobenius property. However, in our case, we do not know whether the first eigenvalue μ_1 of $\mathcal{L}_+$ is negative or not. Due to the presence of the quasilinear term $u\Delta(u\cdot)$, we can not use above simple argument to assert that $\mathcal{L}_+$ satisfies Perron-Frobenius property. Nevertheless, we can still deduce the following result.

Lemma 2.5. *The first eigenvalue μ_1 of $\mathcal{L}_+$ is negative and simple.*

Proof. We have to show that $\mu_1 < 0$ holds and that eigenfunctions corresponding to μ_1 is of constant sign. We argue by contradiction. Suppose that $\mu_1 \geq 0$ holds. Then the fact $0 \in \sigma_{\rm disc}(\mathcal{L}_+)$ implies that $\mu_1 = 0$. Note that $\mathrm{Ker}\mathcal{L}_+ \neq \emptyset$ is the eigenfunction space corresponding to 0. For any $\phi \in \mathrm{Ker}\mathcal{L}_+$, we have that

$$-\Delta\phi - 2u\Delta(u\phi) + \omega\phi - (\Delta u^2 + pu^{p-1})\phi = 0.$$

Since u is a real valued function, we can assume, with no loss of generality, that ϕ is a real valued function as well. Furthermore, we can assume that the positive part $\phi_+ = \max(\phi, 0)$ is not identically zero. Then multiply above equation by ϕ_+. We obtain by integrating by parts that

$$\langle \mathcal{L}_+\phi_+, \phi_+\rangle = 0.$$

That is, $\langle \mathcal{L}_+\phi_+, \phi_+\rangle$ achieves the first eigenvalue 0. Thus ϕ_+ is a combination of eigenfunctions of 0, which implies that ϕ_+ satisfies equation

$$-\Delta\phi_+ - 2u\Delta(u\phi_+) + \omega\phi_+ - (\Delta u^2 + pu^{p-1})\phi_+ = 0. \tag{2.19}$$

We claim that equation (2.19) implies that

$$\phi_+(x) > 0 \qquad \text{for all } x \in \mathbb{R}^N. \tag{2.20}$$

Rewrite equation (2.19) in the form

$$-\Delta\phi_+ - \sum_{i=1}^N b_i(x)\cdot\partial_{x_i}\phi_+ + c(x)\phi_+ = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.21}$$

By Theorem 1.1, both functions

$$b_i(x) \equiv -\frac{4u}{1+2u^2}\partial_{x_i}u, \quad (1 \leq i \leq N) \quad \text{and} \quad c(x) \equiv \frac{\omega - 2u\Delta u - \Delta u^2 - pu^{p-1}}{1+2u^2}$$

are bounded smooth functions. Thus elliptic regularity theory gives us that $\phi_+ \in C^\infty(\mathbb{R}^N)$ holds. Now, by a famous generalized comparison principle for second order elliptic equations due to Serrin (see Theorem 2.10 of Han and Lin [9, Chapter 2]), we deduce from equation (2.21) that (2.20) holds. This proves the claim.

Recall that $\partial_{x_1}u \in \mathrm{Ker}\mathcal{L}_+$. Take $\phi = \partial_{x_1}u = u'(|x|)x_1/|x|$. Since $u'(|x|) < 0$ for $|x| > 0$, we have that $\phi_+(x) \equiv 0$ for any $x \in \mathbb{R}^N$ with $x_1 \geq 0$. We obtain a contradiction to (2.20). Hence we conclude that $\mu_1 < 0$.

Finally, by similar arguments as above, we infer that any eigenfunction corresponding to μ_1 is either positive or negative in $\mathbb{R}^N$. This proves that μ_1 is simple. The proof of Lemma 2.5 is complete. □

Now we are able to prove Propositions 2.2 and 2.3.

Proof of Propositions 2.2 and 2.3. It is enough to prove Proposition 2.3 due to the equivalence. For any function $v \in \mathrm{Ker}\mathcal{L}_+ \cap L^2_{\mathrm{rad}}(\mathbb{R}^N)$, $v \not\equiv 0$, we obtain from above that

$$\int_{\mathbb{R}^N} v\bar{e}_1 \mathrm{d}x = 0$$

holds for any eigenfunction e_1 of $\mathcal{L}_+$ with eigenvalue μ_1. Since e_1 can be chosen strictly positive in $\mathbb{R}^N$, we infer that $v(x) = v(r)$ with $r = |x|$ must change sign for $r > 0$. This proves Proposition 2.3. So follows Proposition 2.2. $\square$

We end this section by showing that $\inf \sigma(\mathcal{L}_+) < 0$ holds for $N = 2$ via direct computations. Precisely, we show that

$$\langle \mathcal{L}_+ u, u\rangle < 0. \tag{2.22}$$

(2.22) follows from a Pohozaev type identity. Since u solves equation (1.2), an elementary calculation gives the following Pohozaev type identity

$$\omega \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = \frac{2}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \mathrm{d}x.$$

Here we used the fact $N = 2$. On the other hand, multiplying equation (1.2) by u and integrating by parts yields

$$\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + 4\int_{\mathbb{R}^2} |u|^2 |\nabla u|^2 \mathrm{d}x + \omega \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = \int_{\mathbb{R}^2} |u|^{p+1} \mathrm{d}x.$$

Recall that

$$\langle \mathcal{L}_+ u, u\rangle = 8\int_{\mathbb{R}^2} |u|^2 |\nabla u|^2 \mathrm{d}x - (p-1)\int_{\mathbb{R}^2} |u|^{p+1} \mathrm{d}x.$$

Combining above three identities, we deduce that

$$\langle \mathcal{L}_+ u, u\rangle = -2\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \frac{(p-1)^2}{p+1}\int_{\mathbb{R}^2} |u|^{p+1} \mathrm{d}x < 0.$$

This proves (2.22). Thus we conclude that $\inf \sigma(\mathcal{L}_+) < 0$ holds for $N = 2$.

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